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# Inverse problems for random differential equations using the collage method for random contraction mappings

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ABSTRACT

In this paper we are concerned with differential equations with random coefficients which will be considered as random fixed point equations of the form $T(\omega, x(\omega)) = x(\omega)$, $\omega \in \Omega$. Here $T : \Omega \times X \to X$ is a random integral operator, $(\Omega, \mathcal{F}, P)$ is a probability space and $X$ is a complete metric space. We consider the following inverse problem for such equations: Given a set of realizations of the fixed point of $T$ (possibly the interpolations of different observational data sets), determine the operator $T$ or the mean value of its random components, as appropriate. We solve the inverse problem for this class of equations by using the collage theorem for contraction mappings.



## 1. Introduction

In this paper, we present a method of solving inverse problems for differential equations with random coefficients using fixed point theory for random contractive operators. The random differential equations will assume the form

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} x(\omega, t) = f(t, \omega, x(\omega, t)), \\ x(\omega, 0) = x_0(\omega), \end{cases} \tag{1}$$

where both the vector field $f$ and the initial condition $x_0$ are random variables defined on an appropriate probability space $(\Omega, \mathcal{F}, P)$ (Such situations comprise the first and third classes of random differential equations as classified by Soong in his classic work [17]). For example, in the case that the vector field in Eq. (1) is polynomial, its coefficients $a_k$ may be considered as random variables. Analogous to the deterministic case, for $X = C([0, T])$ this problem can be reformulated by using the following random integral operator $T : \Omega \times X \to X$:

$$(T_\omega x)(t) = x_0(\omega) + \int_0^t f(s, \omega, x(s))\, \mathrm{d}s. \tag{2}$$

Solutions to (1) are fixed points of (2), that is solution of the equation $T_\omega x = x$. Many papers in the literature deal with such equations for single-valued and set-valued random operators — see, for example [1,7,8,16] and references therein.

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We draw upon results of a recent paper [14] in which the forward and inverse problems of such random fixed point equations were provided in the case that the random operator $T$ is contractive. Our results, including a random collage theorem, are essentially stochastic analogs of classical (deterministic) Banach fixed point theory. In this way, we provide a mathematical basis for the important problem of parameter estimation for random differential equations [9].

## 2. Classical fixed point equations, associated inverse problems and the collage theorem

It is first instructive to recall some basic facts regarding contraction maps that will be used in later sections. In what follows, $(X, d_X)$ denotes a complete metric space. Then $T : X \to X$ is contractive if there exists a $c \in [0, 1)$ such that $d_X(Tx_1, Tx_2) \le c d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. We normally refer to the infimum of all $c$ values as the *contraction factor* of $T$. Under these hypotheses Banach's fixed point theorem states that there exists a unique $\bar{x} \in X$ such that $\bar{x} = T\bar{x}$ and, for any $x \in X$, $d_X(T^n x, \bar{x}) \to 0$ as $n \to \infty$. We now state a formal mathematical *inverse problem* associated with the fixed point equation $x = Tx$ as follows [6]: Given a target element $x$ and an $\epsilon > 0$, find a contraction map $T(\epsilon)$ (perhaps from a suitable family of operators [4]) with fixed point $\bar{x}(\epsilon)$ such that $d_X(x, \bar{x}(\epsilon)) < \epsilon$. If one is able to solve such an inverse problem to arbitrary precision, i.e., $\epsilon \to 0$, then one may identify the target $x$ as the fixed point of a contractive operator $T$ on $X$. In practical applications, it is generally not possible to find such solutions to arbitrary accuracy nor is it even computationally feasible to search for such contraction maps. Instead, one may make use of the following result, a simple consequence of Banach's fixed point theorem.

**Theorem 1** (*"Collage Theorem" [2]*). *Let $(X, d_X)$ be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$, $d_X(x, \bar{x}) \le \frac{1}{1-c} d_X(x, Tx)$, where $\bar{x}$ is the fixed point of $T$.*

Note that the *approximation error* $d_X(x, \bar{x})$ can be controlled by the so-called *collage distance* $d_X(x, Tx)$. Most practical methods of solving such inverse problems, for example, fractal image coding [5,15], search for an operator $T$ for which the collage distance is as small as possible. In other words, they seek an operator $T$ that maps the target $x$ as close as possible to itself. This inverse problem procedure, often referred to as *collage coding*, is most often performed by considering a parametrized family of contraction maps $T_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}^n$, and then minimizing the collage distance $d_X(x, T_\lambda x)$.

In [13], we showed how collage coding could be used to solve inverse problems for systems of DEs having the form

$$\dot{x} = f(t, x), \qquad x(0) = x_0 \tag{3}$$

in the case when the coefficients are polynomial. One begins with the associated Picard integral operator

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))\,\mathrm{d}s \tag{4}$$

which, under certain conditions on $f$, is contractive on an interval $(-\delta, \delta)$, for some $\delta > 0$. For nonautonomous systems of DEs, it is convenient to employ polynomial vector fields, i.e.,

$$f(x) = \sum_{i=1}^{n} a_i x^i. \tag{5}$$

Each vector of coefficients $\mathbf{a} = \{a_1, \dots, a_n\} \in \mathbb{R}^n$ then defines a Picard operator $T_{\mathbf{a}}$. Given a target solution $x(t)$, we now minimize the collage distance $\|x - T_{\mathbf{a}}x\|$. This results in a linear system of equations involving the $a_i$ and possibly $x_0$. We refer the reader to [13] and later works [10–12] for more details on the implementation of the method.

**Example 1.** We consider the damped harmonic oscillator system

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2, \qquad x_1(0) = x_{10} \tag{6}$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = -bx_2 - kx_1, \qquad x_2(0) = x_{20}. \tag{7}$$

To simulate an experiment, we set

$$b = 0.2, k = 0.5, x_{10} = \frac{1}{3} \quad \text{and} \quad x_{20} = \frac{1}{3},$$

and then solve numerically the system of ODEs. For $t \in [0, 30]$, we sample the solutions at 40 uniformly spaced points. Degree-20 polynomials are fitted to the resulting simulated observational data. These two polynomials are our target functions. That is, we seek a Picard operator of the form in (4), with

$$f(x) = \begin{pmatrix} c_1 x_2 \\ c_2 x_2 + c_3 x_1 \end{pmatrix},$$

and with the components of $x_0$ as parameters as well. The result of the process is illustrated in Fig. 1.

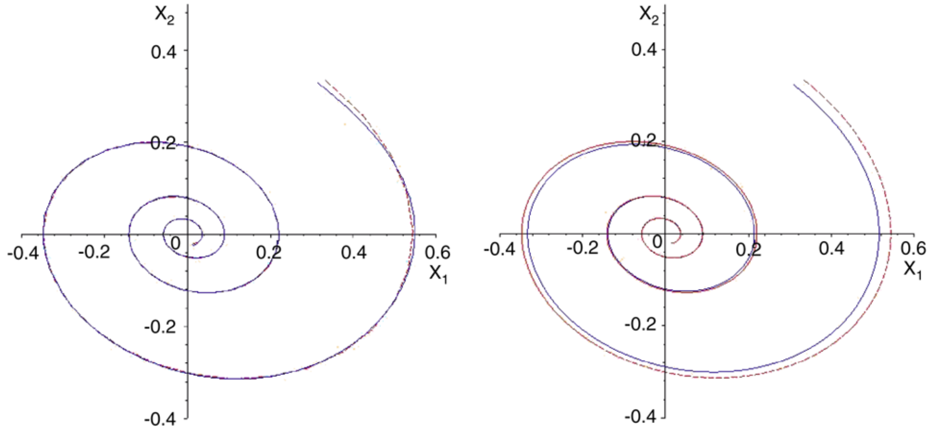


**Fig. 1.** Graphs in phase space for Example 1. (left) The numerical solution (dashed) and the fitted target. (right) The target (dashed) and the fixed point of the resulting minimal-collage Picard operator.

The minimal-collage system to five decimal places is

$$f(x) = \begin{pmatrix} 0.98801x_2 \\ -0.50158x_2 - 0.19431x_1 \end{pmatrix}, \qquad x_0 = \begin{pmatrix} 0.30921 \\ 0.32253 \end{pmatrix}.$$

## 3. Random fixed point equations

As above, let $(X, d_X)$ denote a complete metric space. If $(X, d_X)$ is separable, then it is said to be a *Polish space*. We also let $(\Omega, \mathcal{F}, P)$ be a probability space. We recall that a function $T : \Omega \times X \to X$ is called a *random operator* (in a strict sense, see [3], p. 104) if for any $x \in X$ the function $T(\cdot, x)$ is measurable. The random operator $T$ is said to be continuous/Lipschitz/contractive if, for a.e. $\omega \in \Omega$, we have that $T(\omega, \cdot)$ is continuous/Lipschitz/contractive. A measurable mapping $x : \Omega \to X$ is called a *random fixed point* of the random operator $T$ if $x$ is a solution of the equation

$$T(\omega, x(\omega)) = x(\omega), \quad a.e. \omega \in \Omega. \tag{8}$$

Consider the space $Y$ of all measurable functions $x : \Omega \to X$. If we define the operator $\tilde{T} : Y \to Y$ as $(\tilde{T}y)(\omega) = T(\omega, x(\omega))$ the solutions of this fixed point equation on $Y$ are the solutions of the random fixed point equation $T(\omega, x(\omega)) = x(\omega)$. Suppose that the metric $d_X$ is bounded, that is, $d_X(x_1, x_2) \leq K$ for all $x_1, x_2 \in X$. Then the function $\psi : \Omega \to \mathbb{R}$, $\psi(\omega) = d_X(x_1(\omega), x_2(\omega))$, is an element of $L^1(\Omega)$ for all $x_1, x_2 \in \Omega$. Now define the following function over the space $Y \times Y$, $d_Y(x_1, x_2) = \int_\Omega d_X(x_1(\omega), x_2(\omega))\mathrm{d}\omega = \mathbb{E}(d_X(x_1(\cdot), x_2(\cdot)))$. In [14] we proved that the space $(Y, d_Y)$ is a complete metric space.

**Theorem 2** (*[14]*). *Suppose that*

(i) *for all $x \in Y$ the function $\xi(\omega) := T(\omega, x(\omega))$ belongs to $Y$,*
(ii) *$d_Y(\tilde{T}x_1, \tilde{T}x_2) \leq c d_Y(x_1, x_2)$ with $c < 1$.*

*Then there exists a unique solution of $\tilde{T}\bar{x} = \bar{x}$, that is, $T(\omega, \bar{x}(\omega)) = \bar{x}(\omega)$ for a.e. $\omega \in \Omega$.*

Hypothesis (i) can be avoided if $X$ is a Polish space, in which case the following result holds.

**Theorem 3** (*[7]*). *Let $X$ be a Polish space, that is, a separable complete metric space, and $T : \Omega \times X \to X$ be a mapping such that for each $\omega \in \Omega$ the function $T(\omega, .)$ is $c(\omega)$-Lipschitz and for each $x \in X$ the function $T(., x)$ is measurable. Let $x : \Omega \to X$ be a measurable mapping; then the mapping $\xi : \Omega \to X$ defined by $\xi(\omega) = T(\omega, x(\omega))$ is measurable.*

**Corollary 1.** *Let $T : \Omega \times X \to X$ be a mapping such that for each $\omega \in \Omega$ the function $T(\omega, .)$ is a $c(\omega)$-contraction. Suppose that for each $x \in X$ the function $T(., x)$ is measurable. Then there exists a unique solution of the equation $\tilde{T}\bar{x} = \bar{x}$ that is $T(\omega, \bar{x}(\omega)) = \bar{x}(\omega)$ for a.e. $\omega \in \Omega$.*

The associated inverse problem for random operators can now be formulated as follows: Given a function $\bar{x} : \Omega \to X$ and a family of operators $\tilde{T}_\lambda : Y \to Y$ find $\lambda$ such that $\bar{x}$ is the solution of random fixed point equation $\tilde{T}_\lambda \bar{x} = \bar{x}$, that is, $T_\lambda(\omega, \bar{x}(\omega)) = \bar{x}(\omega)$. As a consequence of the collage and continuity theorems, we have the following.

**Corollary 2** (*Random Collage Theorem*). *Suppose that*

(i) *for all $x \in Y$ the function $\xi(\omega) := T(\omega, x(\omega))$ belongs to $Y$,*
(ii) *$d_Y(\tilde{T}x_1, \tilde{T}x_2) \leq c d_Y(x_1, x_2)$ with $c < 1$.*

*Then for any $x \in Y$, $d_Y(x, \bar{x}) \leq \frac{1}{1-c} d_Y(x, \tilde{T}x)$, where $\bar{x}$ is the fixed point of $\tilde{T}$, that is, $\bar{x}(\omega) := T(\omega, \bar{x}(\omega))$.*

## 4. Inverse problem for random differential equations

For notational convenience, we consider the case of scalar random integral equations, but analogous results can be proved in similar ways for the vector-valued case. Let $(\Omega, \mathcal{F}, P)$ be a given probability space. Let $\phi : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions

1. $|\phi(\omega, t, z_1) - \phi(\omega, t, z_2)| \le K_\phi(\omega)|z_1 - z_2|$
2. $\sup_{t\in[t_0,t_0+\delta]} |\phi(\omega, t, 0)| = \tilde{K}_\phi(\omega)$

a.e. $\omega \in \Omega$, where $K_\phi$ and $\tilde{K}_\phi$ are random variables with known distribution. Given $\alpha \in [0, 1]$, let $K_\alpha$ be defined such that $\Omega_\alpha = \left\{w \in \Omega : K_\phi(w), \tilde{K}_\phi(w), x_0(w) \in [-K_\alpha, K_\alpha]\right\}$ with $P(\Omega_\alpha) \ge \alpha$. Let $M > K_\alpha$ and consider now the space $X = \{x \in C([t_0, t_0 + \delta]) : \|x\|_\infty \le M\}$ endowed with the usual $d_\infty$ metric.

**Proposition 1.** *For a.e. $\omega \in \Omega_\alpha$, let $(T_\omega x)(t) = \int_{t_0}^t \phi(\omega, s, x(s))\mathrm{d}s + x_0(\omega)$, $t \in [t_0, t_0 + \delta]$. If $\delta$ is small enough, then $T_\omega : X \to X$.*

Consider now the operator $\tilde{T} : Y \to Y$ where $Y = \{x : \Omega_\alpha \to X, x \text{ is measurable}\}$ and

$$[(\tilde{T}x)(\omega)](t) = \int_{t_0}^t \phi(\omega, s, [x(\omega)](s))\mathrm{d}s + x_0(\omega) \tag{9}$$

for a.e. $\omega \in \Omega_\alpha$. We use the notation $[x(\omega)]$ to emphasize that for a.e. $\omega \in \Omega_\alpha$, $[x(\omega)]$ is an element of $X$. We have the following result.

**Proposition 2.** *Let $E_\alpha(K_\phi) = \int_{\Omega_\alpha} K_\phi(\omega)\mathrm{d}P(\omega)$. If $\delta E_\alpha(K_\phi) < 1$ then $\tilde{T}$ is a contraction on $Y$. In particular, if $\delta\mathbb{E}(K_\phi) < 1$ then $\tilde{T}$ is a contraction on $Y$.*

By Banach's theorem we have the existence and uniqueness of the solution of the equation $\tilde{T}x = x$. For a.e. $(\omega, t) \in \Omega_\alpha \times [t_0, t_0 + \delta]$ we have

$$x(\omega, t) = \int_{t_0}^t \phi(\omega, s, x(\omega, s))\mathrm{d}s + x_0(\omega). \tag{10}$$

### 4.1. *The inverse problem*

The following example shows how one could solve inverse problems for specific classes of random differential equations by reducing the problem to inverse problems for ordinary differential equations.

**Example 2.** Let us consider the following system of random equations:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X_t = AX_t + B_t, \\ x(0) = x_0, \end{cases} \tag{11}$$

where $X : \mathbb{R} \times \Omega \to \mathbb{R}^n$, $A$ is a (deterministic) matrix of coefficients and $B_t$ is a classical vector Brownian motion. As above, an inverse problem for this kind of equations can be formulated as: given an i.d. sample of observations of $X(t, \omega)$, say $(X(t, \omega_1), \ldots, X(t, \omega_n))$, get an estimation of the matrix $A$. For this purpose, let us take the integral over $\Omega$ of both sides of the previous equation and suppose that $X(t, \omega)$ is sufficiently regular; recalling that $B_t \sim \mathcal{N}(0, t)$, we have

$$\int_\Omega \frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}P(\omega) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(X(t, \cdot)) = A\mathbb{E}(X(t, \cdot)). \tag{12}$$

This is a deterministic differential equation in $\mathbb{E}(X(t, \cdot))$. From the sample of observations of $X(t, \omega)$ we can then get an estimation of $\mathbb{E}(X(t, \cdot))$ and then use the approach developed for deterministic differential equations to solve the inverse problem for $A$. As a numerical example, let us consider the first-order system

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}x_t &= a_1 x_t + a_2 y_t + b_t \\ \frac{\mathrm{d}}{\mathrm{d}t}y_t &= b_1 x_t + b_2 y_t + c_t. \end{aligned}$$

Setting $a_1 = 0.8$, $a_2 = -0.7$, $b_1 = 0.9$, $b_2 = 0.6$, $x_0 = 1.2$, and $y_0 = 1$,we construct observational data values for $x_t$ and $y_t$ for $t_i = \frac{i}{N}$, $1 \le i \le N$, for various values of $N$. For each of $M$ data sets, different pairs of Brownian motion are simulated for $b_t$ and $c_t$. Fig. 2 presents several plots of $b_t$ and $c_t$ for $N = 100$. In Fig. 3, we present some plots of our generated $x_t$ and $y_t$, as

**Table 1**
Minimal collage distance parameters for different $N$ and $M$

| $N$ | $M$ | $a_1$ | $a_2$ | $b_1$ | $b_2$ |
|---|---|---|---|---|---|
| 10 | 10 | 0.6893 | −0.4456 | 0.8753 | 0.3813 |
| 10 | 100 | 0.8617 | −0.6617 | 0.7834 | 0.6269 |
| 100 | 10 | 0.8499 | −0.5981 | 0.9055 | 0.5757 |
| 100 | 100 | 0.6842 | −0.6163 | 0.9319 | 0.5823 |

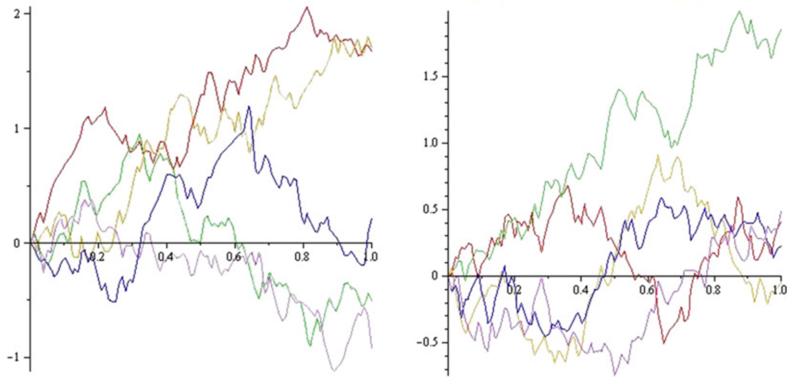


**Fig. 2.** Example plots of $b_t$ and $c_t$ for $N = 100$.

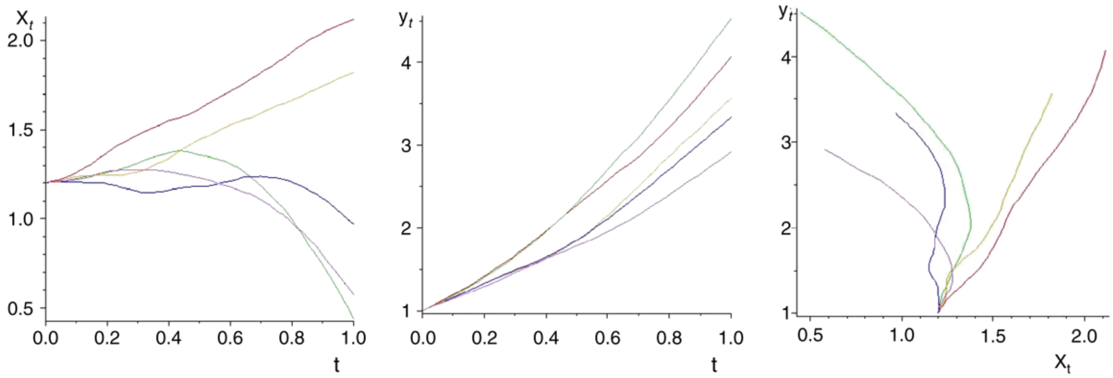


**Fig. 3.** Example plots of $x_t$, $y_t$, and $x_t$ versus $y_t$ for $N = 100$.

well as phase portraits for $x_t$ versus $y_t$. For each sample time, we construct the mean of the observed data values, $x^*_{t_i}$ and $y^*_{t_i}$, $1 \le i \le N$. We minimize the squared collage distances

$$\Delta_x^2 = \frac{1}{N}\sum_{i=1}^{N}\left(x^*_{t_i} - x_0 - \frac{1}{N}\sum_{j=1}^{i}\left(a_1 x^*_{t_j} + a_2 y^*_{t_j}\right)\right)^2$$

and

$$\Delta_y^2 = \frac{1}{N}\sum_{i=1}^{N}\left(y^*_{t_i} - y_0 - \frac{1}{N}\sum_{j=1}^{i}\left(b_1 x^*_{t_j} + b_2 y^*_{t_j}\right)\right)^2$$

to determine the minimal collage parameters $a_1$, $a_2$, $b_1$, and $b_2$. The results of the process are summarized in Table 1.

In what follows, we suppose that $\phi(\omega, t, z)$ has the following polynomial form in $t$ and $z$:

$$\phi(\omega, t, z) = a_0(\omega) + a_1(\omega)t + a_2(\omega)z + a_3(\omega)t^2 + a_4(\omega)tz + a_5(\omega)z^2 + \cdots. \quad (13)$$

Suppose that $x_0$ and each $a_i$ are random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$. Let $\mu$ and $\sigma^2$ be the unknown mean and variance of $x_0$ and $\nu_i$ and $\sigma_i^2$ be the unknown means and variances of $a_i$. Given data from independent realizations $x(\omega_j, t)$, $j = 1, \ldots, N$, of the random variable $x$, the fixed point of $T$, we wish to recover the means and variances.

**Table 2**
Distributions used in the inverse problem of Example 2

| Label | True Values | | | |
|---|---|---|---|---|
| | $a_0$ | $a_1$ | $a_2$ | $x_0$ |
| 1 | $\mathcal{N}(1, 0.04)$ | $\mathcal{N}(0.7, 0.04)$ | $\mathcal{N}(0.3, 0.04)$ | $\mathcal{N}(0.4, 0.09)$ |
| 2 | $\mathcal{N}(2, 0.09)$ | $\mathcal{N}(0.5, 0.09)$ | $\mathcal{N}(0.4, 0.09)$ | $\mathcal{N}(0.5, 0.04)$ |
| 3 | $\chi^2(6)$ | $\chi^2(5)$ | $\chi^2(6)$ | $\chi^2(4)$ |

**Table 3**
Results for the inverse problem of Example 1

| Label | $N$ | Minimal collage values | | | |
|---|---|---|---|---|---|
| | | $a_0$ | $a_1$ | $a_2$ | $x_0$ |
| 1 | 10 | (0.98296, 0.06556) | (0.68003, 0.02753) | (0.30730, 0.02874) | (0.37880, 0.02048) |
| 1 | 100 | (0.97445, 0.04477) | (0.69110, 0.03227) | (0.34502, 0.03363) | (0.41115, 0.08420) |
| 1 | 1000 | (0.99815, 0.03712) | (0.69811, 0.03819) | (0.29583, 0.04097) | (0.38900, 0.08953) |
| 2 | 10 | (1.97448, 0.14749) | (0.47005, 0.06195) | (0.41095, 0.06466) | (0.48587, 0.00910) |
| 2 | 100 | (1.96168, 0.10073) | (0.48664, 0.07262) | (0.46753, 0.07567) | (0.50743, 0.03742) |
| 2 | 1000 | (1.99723, 0.08374) | (0.49716, 0.08593) | (0.39374, 0.09217) | (0.49267, 0.03979) |
| 3 | 10 | (6.13277, 9.80128) | (5.89477, 10.90187) | (3.48752, 3.44609) | (3.41455, 6.00164) |
| 3 | 100 | (5.98742, 11.46678) | (5.02497, 9.23188) | (5.61510, 10.33998) | (4.55106, 9.91036) |
| 3 | 1000 | (6.15257, 12.39693) | (4.71144, 8.28957) | (5.96555, 11.05111) | (3.93438, 7.83029) |

The first column indicates the distributions used from Table 2. $N$ is the number of realizations, and the final four columns give the (mean, variance) obtained via collage coding for each parameter.

**Table 4**
Distributions used in the inverse problem of Example 3

| Label | True values | | | |
|---|---|---|---|---|
| | $a_0$ | $a_1$ | $a_5$ | $x_0$ |
| 1 | $\mathcal{N}(1, 0.04)$ | $\mathcal{N}(0.7, 0.04)$ | $\mathcal{N}(0.3, 0.04)$ | $\mathcal{N}(0.4, 0.09)$ |
| 2 | $\mathcal{N}(2, 0.09)$ | $\mathcal{N}(0.5, 0.09)$ | $\mathcal{N}(0.4, 0.09)$ | $\mathcal{N}(0.5, 0.04)$ |

Each realization $x(\omega_j, t), j = 1, \dots, N$, of the random variable is the solution of a fixed point equation

$$x(\omega_j, t) = \int_0^t \phi(\omega_j, s, x(\omega_j, s))\mathrm{d}s + x_0(\omega_j) = \int_0^t \Big[a_0(\omega_j) + a_1(\omega_j)s + a_2(\omega_j)x(\omega_j, s) + a_3(\omega_j)s^2 + a_4(\omega_j)sx(\omega_j, s) + a_5(\omega_j)(x(\omega_j, s))^2 + \cdots\Big]\mathrm{d}s + x_0(\omega_j).$$

Thus, for each target function $x(\omega_j, t)$, we can find samples of realizations for $x_0(\omega_j)$ and $a_i(\omega_j)$ via the collage coding method for polynomial deterministic integral equations outlined in Section 2. Upon treating each realization, we will have determined $x_0(\omega_j)$ and $a_i(\omega_j)$, $i = 1, \dots, M$, $j = 1, \dots, N$. We then construct the approximations

$$\mu \approx \mu_N = \frac{1}{N}\sum_{j=1}^{N} x_0(\omega_j) \quad \text{and} \quad \nu_i \approx (\nu_i)_N = \frac{1}{N}\sum_{j=1}^{N} a_i(\omega_j), \tag{14}$$

where we note that results obtained from collage coding each realization are independent. Using our approximations of the means, we can also calculate that

$$\sigma^2 \approx \sigma_N^2 = \frac{1}{N-1}\sum_{j=1}^{N}(x_0(\omega_j) - \mu_N)^2, \qquad \sigma_i^2 \approx (\sigma_i)_N^2 = \frac{1}{N-1}\sum_{j=1}^{N}(a_i(\omega_j) - (\nu_i)_N)^2.$$

**Example 3.** We consider the linear case, $\phi(\omega, t, z) = a_0(\omega) + a_1(\omega)t + a_2(\omega)z$. The realizations are calculated by solving numerically the related differential equation, sampling the solution at 10 uniformly distributed points, and fitting a sixth-degree polynomial $x(\omega_j, t)$ to the data. Fig. 4 illustrates some of the realizations. Table 2 lists the distributions from which the parameters that generate each realization are selected. The results of the preceding approach to the inverse problem are presented in Table 3.

We include an example wherein the parameters are selected from $\chi^2$ distributions. This example shows that we can avoid the technical details of Section 4 that define the maximal allowed value of $\delta$ by instead just choosing $\delta$ very small. In this example, we pick $\delta = 0.1$.

**Example 4.** We suppose that $\phi(\omega, t, z)$ is quadratic in z, namely $\phi(\omega, s, z) = a_0(\omega) + a_1(\omega)z + a_5(\omega)z^2$. The realizations are calculated by solving numerically the related differential equation, sampling the solution at 10 uniformly distributed

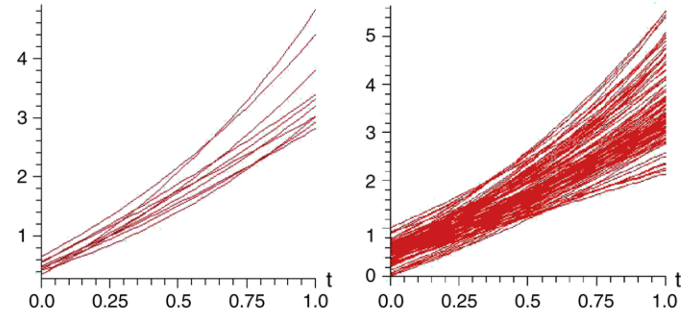


**Fig. 4.** Graphs for Example 1, with linear $\phi$. (left to right) 10 realizations, and 100 realizations.

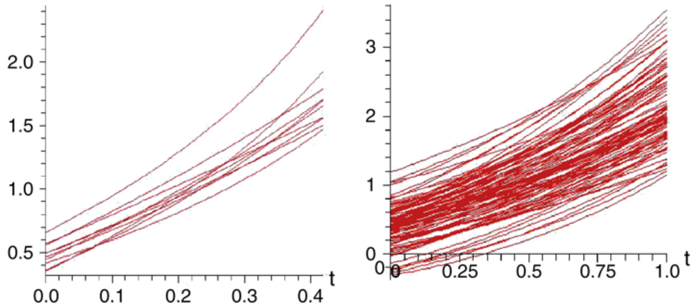


**Fig. 5.** Graphs for Example 2, with quadratic $\phi$. (left to right) 10 realizations, and 100 realizations.

**Table 5**
Results for the inverse problem of Example 2

| Label | $N$ | Minimal collage values | | | |
|---|---|---|---|---|---|
| | | $a_0$ | $a_1$ | $a_5$ | $x_0$ |
| 1 | 10 | (0.99165, 0.06671) | (0.67052, 0.02730) | (0.30981, 0.02977) | (0.37881, 0.02022) |
| 1 | 100 | (0.97774, 0.04479) | (0.68779, 0.03163) | (0.34587, 0.03401) | (0.41102, 0.08408) |
| 1 | 1000 | (1.00016, 0.03738) | (0.69583, 0.03793) | (0.29648, 0.04137) | (0.38891, 0.08947) |
| 2 | 10 | (1.97450, 0.14737) | (0.46999, 0.06195) | (0.41098, 0.06356) | (0.48587, 0.00910) |
| 2 | 100 | (1.96166, 0.10061) | (0.48657, 0.07280) | (0.46764, 0.07581) | (0.50743, 0.03742) |
| 2 | 1000 | (1.99693, 0.08401) | (0.49815, 0.08796) | (0.39301, 0.09341) | (0.49267, 0.03979) |

The first column indicates the distribution from Table 4 from which realizations are generated. $N$ is the number of realizations, and the final four columns give the (mean, variance) obtained via collage coding for each parameter.

**Table 6**
Results for the random damped oscillator inverse problem of Example 4

| $N$ | Minimal collage values | | | |
|---|---|---|---|---|
| | $b$ | $k$ | $x_{10}$ | $x_{20}$ |
| 10 | (0.28140, 0.02128) | (0.59233, 0.02311) | (0.33273, 0.00532) | (0.41371, 0.00391) |
| 30 | (0.22405, 0.02055) | (0.51808, 0.02302) | (0.30006, 0.01300) | (0.37182, 0.00709) |
| 100 | (0.21539, 0.02310) | (0.50649, 0.02081) | (0.34361, 0.00928) | (0.32786, 0.01190) |

The results for the random variables are given as (mean, variance).

points, and fitting the a polynomial $x(\omega_j, t)$ to the data. Fig. 5 presents some of the realizations. Table 4 lists the distributions used for the parameters, and the collage coding results are presented in Table 5. Although the theoretical presentation of the above sections deals with a single equation, the results extend naturally to systems. In the following final example, we return to the damped oscillator problem from Example 1.

**Example 5.** We replace the constant coefficients of Example 1 by random variables. In order to generate realizations, we assume that the coefficients $b$ and $k$ in the above equations, as well as the initial conditions, are random variables selected from a chosen distribution: $\mathcal{N}(0.2, 0.02)$, $\mathcal{N}(0.5, 0.02)$, $\mathcal{N}(\frac{1}{3}, 0.01)$, and $\mathcal{N}(\frac{1}{3}, 0.01)$, respectively. We generate $N$ realizations, and fit a polynomial target to uniformly sampled data points. Next, we collage code each target (as in Example 1), and finally calculate the mean parameter values and corresponding mean operator's fixed point. Results are presented in Table 6. Fig. 6 presents some visual results. In Fig. 6, the left pictures show graphs in phase space of both the realizations

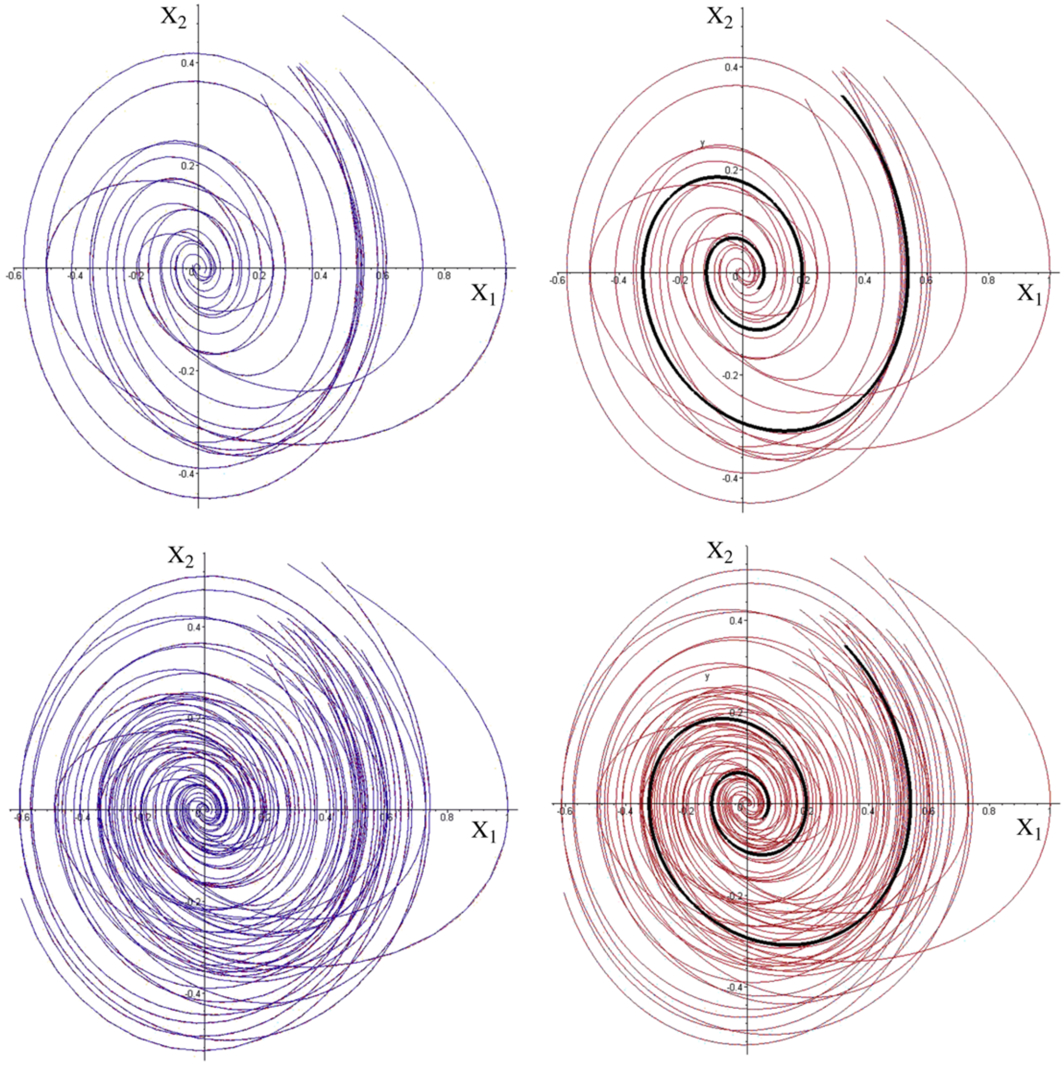


**Fig. 6.** Graphs for Example 4. (top) 10 realizations, (left) realizations and targets, (right) targets, collage coding fixed points, and fixed point of the operator defined by the mean parameter values (thicker curve). (bottom) Similar results for 30 realizations.

and the fitted polynomials; these orbits are coincident at the resolution of the picture, but there are in fact slight errors. The graphs on the right show the target polynomials along with the fixed points calculated via collage coding for deterministic integral equations; once again, at the resolution of the picture the orbits appear to be coincident. The thicker orbit in each graph on the right is the fixed point of the operator defined by mean parameter values.

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